Parallel and orthogonal stimulus in ultradiluted neural networks

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(Received 20 July 2006; revised manuscript received 23 August 2006; published 27 October 2006)

Extending a model due to Derrida, Gardner, and Zippelius, we have studied the recognition ability of an extreme and asymmetrically diluted version of the Hopfield model for associative memory by including the effect of a stimulus in the dynamics of the system. We obtain exact results for the dynamic evolution of the average network superposition. The stimulus field was considered as proportional to the overlapping of the state of the system with a particular stimulated pattern. Two situations were analyzed, namely, the external stimulus acting on the initialization pattern (parallel stimulus) and the external stimulus acting on a pattern orthogonal to the initialization one (orthogonal stimulus). In both cases, we obtained the complete phase diagram in the parameter space composed of the stimulus field, thermal noise, and network capacity. Our results show that the system improves its recognition ability for parallel stimulus. For orthogonal stimulus two recognition phases emerge with the system locking at the initialization or stimulated pattern. We confront our analytical results with numerical simulations for the noiseless case $T=0$.

DOI: 10.1103/PhysRevE.74.046117 PACS number(s): 84.35.+i, 07.05.Mh, 05.70.Fh

I. INTRODUCTION

In the last years, many specific biological characteristics of living neurons have been incorporated to the theoretical neural network models, in particular to the Hopfield model of pattern recognition [1]. The main interest is devoted to describe more realistically the properties of natural neural networks, although many extensions of the traditional modelling have indeed led to the emergence of new phenomena [2–4]. In particular, several works have been developed aiming an increase in the recognition ability of learning patterns or even an enhancement of the associated basin of attraction which may result in more robust networks [5,6]. In this direction, an important biological element to be introduced in neural network models is related to external stimulus. Biologically, whenever a stimulus reaches our mind, even through an image, a sound or a sensation, a set of neurons is activated which stimulates or inhibits each other in order to achieve a given conscious state, i.e., a network pattern. Once the stimulus is turned off, the neurons that participated in this process may or may not return to their original state. If the stimulus is persistent, the set of activated neurons naturally strengthens their interactions, thus increasing the response efficiency to this stimulus.

Several proposals have been presented in the literature taking into account the effect of external stimulus acting on neural networks using both analytical and numerical techniques. One of the main lines of works considers the action of an external field as the network stimulus. Amit, Gutfreund, and Sompolinsky [7,8] analyzed the effect of a static field conjugated to one or several patterns on the thermodynamic behavior of the Hopfield model. The main result for the particular case of a single stimulated pattern was an increase of the recognition ability associated with this pattern, i.e., the network with N neurons acquires the ability to recover the pattern even when the number of memorized patterns p exceeds the limit $\alpha=p/N\simeq 0.14$ typical of the zero field case. However, they found that, when the stimulus is over several patterns, there is an increase of internal noise and the resulting increase of the recognition ability in insignificant, decaying with the number stimulated patterns. For example, a stimulus over five or six patterns already generates enough internal noise to strongly restrict the recognition ability.

In 1989, Engel *et al.* [9] proposed the use of external fields parallel to the initial configuration to study the pattern recognition ability of the Hopfield model. The correlation between the external field and the initialization state allowed one to combine the internal information stored in the synapses with the input stimulus. They compared analytical and numerical results demonstrating that such external stimulus also allows for pattern recognition even when the number of stored patterns exceeds the critical value for the zero field Hopfield model. The interdependence of the state superposition, external stimulus and the initial condition was studied, revealing a first order transition between the disordered and recognition phases, for values of the initial superposition above a critical value.

The use of external neural stimulus, parallel to the initial configuration, was later extended to the diluted Hopfield model [10]. In this case, a dynamic equation for the time evolution of the system superposition was obtained as a function of the parameter that controls the intensity of the external stimulus. By investigating the basin of attraction of the stimulated pattern, they found an increased stability of the memory states and an additional decrease of spurious states for a narrow range of values for the stimulus intensity. In this region, they found that recognition could be achieved even when the initial configuration was outside the zero-field basin of attraction of the initial pattern. However, when the number of memorized patterns is large, the stability of the spurious states increases with the external field and therefore, such stimulus becomes unable to promote an enhancement of the network recognition ability in this regime.

Amit, Parisi, and Nicolis [11] also investigated a model in which an external field coupled to one of the patterns is used to stimulate the network. They treated analytically this model within the mean field approximation and compared their results with numerical simulations. This model differs from the one studied by Amit, Gutfreund, and Sompolinsky [7] by the presence of noise in the external field, which is generated by a predetermined error distribution. They found that the level of memory upload above which the network stops to act as a model of associative recognition is determined by the stimulus intensity at the initial state. Further, they identified that, for any noise distribution and irrespective to the number of patterns stored, there is a critical field value above which the network completely corrects the information contained in the stimulus. More recent studies of hysteresis in driven neural networks under static [12] and dynamic [13] external fields revealed a rich behavior with a maximum hysteresis loop area for a nonzero periodic field intensity.

In the present work, we study the Derrida-Gardner-Zippelius (DGZ) model [14], which consists of an ultradiluted Hopfield neural network, under the action of a stimulus over one of the stored patterns which is proportional to the network state superposition with the stimulated memory pattern. We will distinguish two possible cases. In the first one, the stimulated pattern will be parallel to the pattern strongly correlated to the initial state. In such case, the convergency of the system state to the pattern to be recognized will be favored once the stimulus already brings information about such pattern. Consequently, the basin of attraction of the stimulated pattern is expected to be increased and the recognition ability improved. In the second case, we will consider that the stimulus acts on a stored pattern that is orthogonal to the pattern strongly correlated with the initial state. In such case, the initial state influences but not fully determines the network dynamics. The competition between the basins of attraction of the stimulated and initial patterns may lead to distinct final states associated with either basin. We will obtain the dynamical equation for the proposed model analytically and report the phase diagram of both cases in the presence of internal an thermal noise. Numerical simulations for the zero temperature limit will be provided and compared with the analytical results.

II. ULTRADILUTED NEURAL NETWORK: DGZ MODEL

An important contribution in the Hopfield model was proposed by Derrida, Gardner, and Zippelius (DGZ) in 1987 [14]. They considered the introduction of new biological ingredients such as dilution and asymmetry in the synaptic connections. The motivation of their work was due to the innumerable biological indications that real neural networks do not have all their elements hardwired between themselves. In this way, each neuron is connected only to a reduced number of other neurons (about 10^4 neurons), that is very inferior to the total number of elements of the network (about 10^{11} neurons).

In the DGZ model, each neuron is considered as an Ising spin with two possible states: an up position or a down position depending on whether the neuron has, or has not, fired an electrochemical signal. The state of the neurons in the network is defined by the state vector

$$\vec{S} = |S_1, S_2, \ldots, S_N\rangle. \tag{1}$$

Each neuron is connected to about C other neurons, where the connectivity degree (C) is very low compared to the connectivity in the original Hopfield model (N). The extreme dilution condition ($1 \ll C \ll \ln N$) is taken. The influence that the ith neuron exerts on the jth neuron is given by the Hebb rule

$$J_{ij} = C_{ij} \sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu, \tag{2}$$

where ξ_i^μ are independent random variables assuming values ± 1 with the same probability, representing the state of ith neuron corresponding to the stored pattern with index μ ($\mu = 1, 2, \ldots, p$). C_{ij}'s are random variables chosen according to the following distribution:

$$\rho(C_{ij}) = \frac{C}{N}\delta(C_{ij} - 1) + \left(1 - \frac{C}{N}\right)\delta(C_{ij}). \tag{3}$$

We stress that C_{ij} is not necessarily symmetrical. The time evolution of the network is governed by a synchronous stochastic dynamical rule with $S_i(t+1) = \pm 1$ with probability

$$\Pr[S_i(t+1)] = \frac{1}{2}[1 + S_i(t+1)\tanh \beta_0 h_i(t)], \tag{4}$$

where $\beta_0 = 1/T_0$ measures the inverse of the stochastic noise level of the network, which differs from the static noise (referring to the number of stored patterns). However, both tend to let the network recognition unstable. The h_i denotes the Hopfield post synaptic potential

$$h_i(t) = \sum_{j=1}^{N} J_{ij} S_j(t). \tag{5}$$

An other important parameter in this model is the macroscopic overlap between the state of system at a given time t and the stored memories, defined as

$$m_\mu(t) = \frac{1}{N}\sum_{i=1}^{N} \xi_i^\mu S_i(t). \tag{6}$$

Considering the time evolution of the system for the case where the initial configuration is exactly one of the stored patterns [i.e., $S_i(0) = \xi_i^{\mu=1}$], it is possible to obtain the recurrence equation for the overlap between the state of system and the pattern $\mu = 1$ that is exact in the limit of extreme dilution which is given by [14]

$$m(t+1) = \int_{-\infty}^{+\infty} Dz \tanh\left[\frac{m(t) + z\sqrt{\alpha}}{T}\right], \tag{7}$$

where $Dz \equiv \frac{dz e^{-z^2/2}}{\sqrt{2\pi}}$, the storage capacity for the diluted model is defined as $\alpha = p/C$ and the reduced temperature corresponds to $T = T_0/C$.

The definition $\alpha = p/C$ is according with the usual Hopfield model, i.e., in the fully connected network having $C = N$, the storage capacity becomes $\alpha = p/N$. In the long time limit, the system converges to a fixed point. In the case $\alpha = 0$ one finds $T^* = 1$ for the critical temperature above which the network loses its recognition ability. In the absence of thermal noise ($T = 0$), the maximal value of the storage capacity for which the network still presents associative memory is $\alpha_c(T=0)=\frac{2}{\pi}\simeq 0.6366$. This value is greater than the one found in the original Hopfield model ($\alpha_c \simeq 0.14$). In the ultradiluted regime, the transition to the nonrecognition phase occurs continuously, in contrast to of the fully connected Hopfield model, where the transition is of first order.

III. DGZ MODEL WITH STIMULATED PATTERNS

In this section, we introduce and analyze the pattern recognition ability of an extension of the DGZ model including the influence of a stimulus that acts on one of the stored patterns. This task will be employed by including an additive term to the local field while keeping all other parameters unchanged. In this form, the local field assumes the form

$$h_i(t) = h_i^H(t) + h_i^S(t), \tag{8}$$

where $h_i^H(t)$ is the usual local field of the Hopfield model, as defined in the previous section. Here, we will introduce the local field representing a stimulus over a given stored pattern as

$$h_i^S(t) = h_0 m_\nu(t) \xi_i^\nu, \tag{9}$$

where ξ^ν is the stimulated pattern, $m_\nu(t)$ is the state superposition (macroscopic overlap) with the stimulated pattern at time t and h_0 is a parameter to control the amplitude of the stimulus. By considering the stimulus local field proportional to the product of the state superposition with the stimulated pattern, we impose that it will only act when the network state is correlated to the stimulated pattern.

The above proposed stimulus field acts as a feedback effect from a specific pattern that is kept imposed during the retrieval dynamics. In this sense, it is a persistent stimulus. In order to understand how this effect can be produced in practice, one shall note that the total local field can be written as

$$h_i(t) = \sum_{j=1}^{N} \left(J_{ij} + \frac{h_0}{N} \xi_i^\nu \xi_j^\nu \right) S_j(t). \tag{10}$$

Therefore, the persistent effect of the stimulus field can be introduced during the training process by defining an effective biased Hebb rule for the coupling between neurons i and j in the form

$$J_{ij}^{\text{eff}} = C_{ij} \sum_{\mu \neq \nu} \xi_i^\mu \xi_j^\mu + \left(C_{ij} + \frac{h_0}{N} \right) \xi_i^\nu \xi_j^\nu. \tag{11}$$

Once the network is trained with the above rule, the bias towards the pattern ν will be kept during the entire retrieval dynamics. From the biological point of view, biased training reflects the natural tendency that some unusual (e.g., traumatic) experiences may have a stronger influence in the formation of the neuron synapses than the regular daily experiences.

In what follows, we will analyze two distinct cases. First, we will consider the initialization pattern strongly correlated to the stimulated pattern while uncorrelated with all others stored patterns. Secondly, the initialization pattern will be taken as weakly correlated to the stimulated pattern but strongly correlated to an orthogonal stored pattern. In the first case the stimulus will act favoring the recognition of the initialization pattern while, in the second case, it will force the dynamics towards a pattern far from the initialization one.

A. Stimulus parallel to the initialization pattern

Considering the initial state having a single superposition with one of the stored patterns, namely, $m_\mu(t=0)=m(0)\delta_{\mu,\nu}$, all state superpositions with $\mu \neq \nu$ will remain null according to the dynamic equations. Under the presence of a stimulus field acting on the pattern ξ^ν, the time evolution of the state superposition m_ν in the ultradilution limit can be shown to obey

$$m(t+1) = \int_{-\infty}^{+\infty} Dz \tanh\left(\frac{m(t)(h+1) + z\sqrt{\alpha}}{T} \right), \tag{12}$$

where the subscript ν was omitted for simplicity and $h = h_0/C$ is the reduced amplitude of the stimulus field. The case $h=0$ recovers the usual dynamic equation of the DGZ model. At large times, the system achieves an asymptotic stationary state with $m(t)=m$ for all t.

We first analyze the simple case of $\alpha=0$ which represents that a finite number of patterns have been stored by the network. In this case, the asymptotic superposition will be given by

$$m = \tanh\left(\frac{m(h+1)}{T} \right). \tag{13}$$

Therefore, the stimulus field acts only by rescaling the thermal noise. The critical temperature above which the network loses its recognition ability will be given now by $T_c=h+1$, i.e., the stimulus increases the stability of the network against thermal noise.

For the case of an infinite number of patterns stored by the network, while keeping α finite, the stationary state superposition in the absence of thermal noise ($T=0$) can be shown to be given by

$$m = \text{erf}\left(\frac{m(h+1)}{\sqrt{2\alpha}} \right), \tag{14}$$

where $\text{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2} du$ is the error function. In Fig. 1 we show the stationary state superposition as a function of h and α for $T=0$ and the initial configuration strongly correlated to the stimulated pattern. Notice that the stimulus field increases the critical capacity above which the network loses its recognition ability. For $T=0$ the critical capacity is given by $\alpha_c=\frac{2}{\pi}(h+1)^2$.

For the general case of finite temperature and network capacity, a simple analytic expression for the critical surface in the space $T \times \alpha \times h$ could not be derived from the stationary solution of Eq. (12). In Fig. 2, we report the full phase diagram as obtained numerically. It clearly shows that the recognition ability is enhanced by the stimulus field with the convergence to the desired pattern becoming more stable against thermal noise and network capacity.

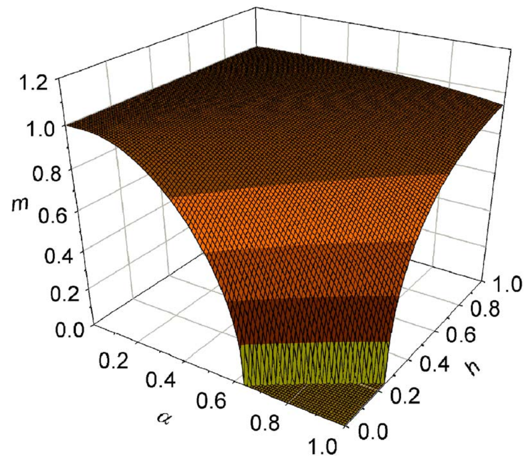

FIG. 1. (Color online) Stationary state superposition m as a function of h and α for $T=0$ when the initial configuration is strongly correlated to the recognition pattern, which is also the stimulated one. The stimulus field increases the critical capacity above which the network loses its recognition ability.

B. Stimulus orthogonal to the initialization pattern

In this section, we consider that the initial state has superposition with two of the stored patterns, i.e., $m_\mu(0) = m_\nu(0)\delta_{\mu,\nu} + m_\delta(0)\delta_{\mu,\delta}$. The initial state will be taken as strongly correlated to the pattern ξ_ν such that $m_\nu(0) \lesssim 1$, but only weakly correlated to the pattern ξ_δ for which $m_\delta(0) \ll 1$. Therefore, in the absence of stimulus the network naturally converges to a state close to ξ_ν whenever pattern recognition takes place. When a stimulus is applied to the pattern ξ_ν the network dynamics is similar to the one described in the previous section. Here, we will assume that the network is stimulated to recognize the pattern ξ_δ which is only weakly correlated to the initialization state. In this case, the network will be forced towards a pattern that is far from the one initially presented to it.

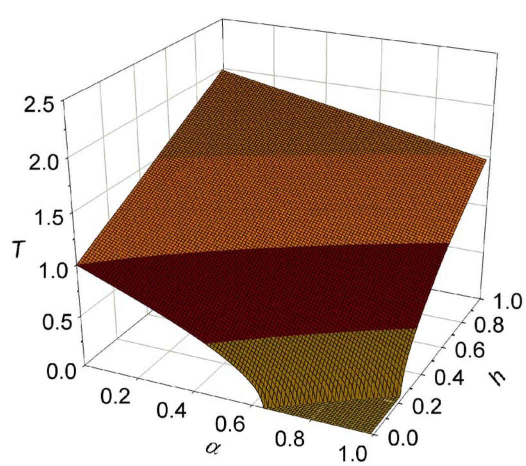

FIG. 2. (Color online) Phase diagram in the space of parameters $T\times\alpha\times h$ for the stimulus field parallel to the pattern strongly correlated to the initialization configuration. Below the critical surface the network retains a finite superposition. The recognition ability increases with increasing stimulus field.

Following the same procedure to deal with the standard DGZ model in the ultradiluted limit, the basic equations to analyze the dynamics of the state superposition with the presented and stimulated patterns can be analytically obtained. In the present case, one results with a pair of coupled equations for the superpositions $m_\nu(t+1)$ and $m_\delta(t+1)$ given by

$$m_\nu(t+1) = \int_{-\infty}^{+\infty} Dz \tanh\left(\frac{m_\nu(t) + z\sqrt{m_\delta^2(t)(1+h)^2 + \alpha}}{T}\right), \tag{15}$$

$$m_\delta(t+1) = \int_{-\infty}^{+\infty} Dz \tanh\left(\frac{m_\delta(t)(h+1) + z\sqrt{m_\nu^2(t) + \alpha}}{T}\right). \tag{16}$$

Notice that both equations are exact and become identical in the absence of the stimulus field. We start analyzing the stationary solution of the above equations for the case of a finite number of stored patterns ($\alpha=0$). In what follows, we will consider $m_\nu(0)=0.99$ and $m_\delta(0)=0.01$. In Fig. 3 we show both state superpositions m_ν and m_δ in the stationary asymptotic regime as functions of T and h. We found that for small values of the stimulus field, there is a discontinuous transition from a state superposed to the pattern ν ($m_\nu \neq 0$ and $m_\delta=0$) to a state superposed to the stimulated pattern δ ($m_\nu=0$ and $m_\delta \neq 0$). Therefore, the presented pattern remains stable in the presence of a weak stimulus towards a different pattern.

At $T=0$ the presented pattern loses its stability for stimulus fields larger than $h_c \simeq 0.279$, with the critical field depending on the initial superposition with the stimulated pattern. For stimulus fields slightly above this critical value, a reentrant behavior was observed. At very low temperatures, the network is driven towards the stimulated pattern even when the network initial state is almost orthogonal to it. In this regime the basin of attraction of the stimulated pattern surpasses that of the initialization pattern. As the temperature is increased, these two basins have their range reduced. The initialization pattern turns out to be the dynamical attractor on a finite temperature range as these two basins become nonoverlapping. Further increasing the temperature makes the initialization pattern unstable and the network is again driven towards the stimulated pattern. Finally, at higher temperatures, all patterns become unstable and the network loses its recognition ability. Such reentrant behavior reflects, therefore, the competing effects between the recognition dynamics and the stimulus field as well as the different role played by the noise on each term. It is important to stress that such reentrant behavior is not directly related to a hysteresis phenomenon. It was previously reported to appear in neural networks trained with noisy data whose origin was also related to competing tendencies in structuring the basins of attraction for the stored patterns [15]. Reentrant behavior is commonly observed in several other physical systems, particularly in spin systems with competing interactions [16,17]. In the present case, the whole phase with $m_\nu \neq 0$ is the region in the parameter space at which the initial state remains stable even in the presence of an orthogonal stimulus. It shall be

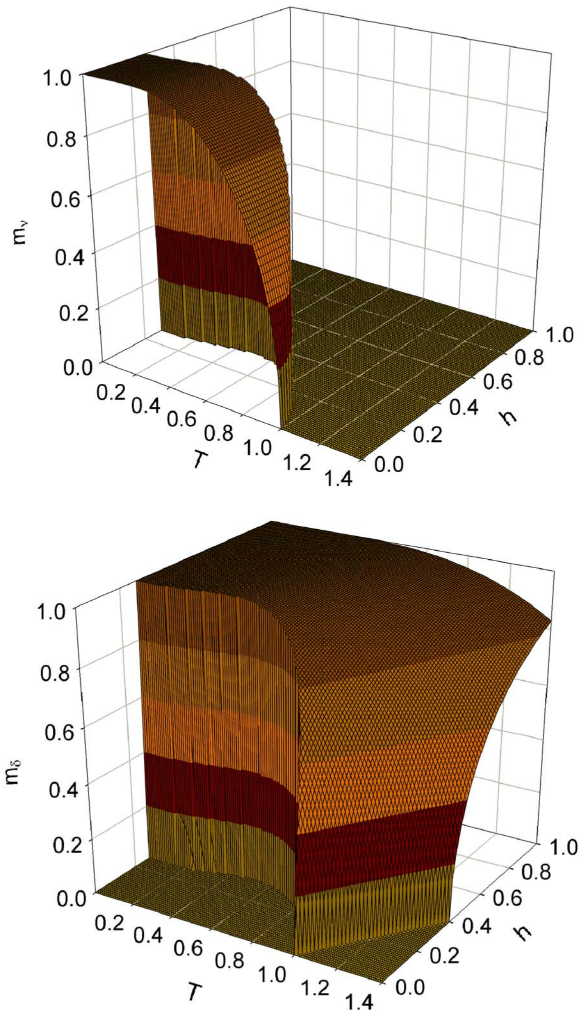

FIG. 3. (Color online) State superpositions m_ν and m_δ in the stationary asymptotic regime as functions of T and h for the case of orthogonal stimulus and a finite number of stored patterns ($\alpha=0$). The initial superpositions were taken as $m_\nu(0)=0.99$ and $m_\delta(0)=0.01$. For low temperatures and stimulus field, the network retains its ability to recognize the pattern strongly correlated to the initial state. As the field increases, the stimulated pattern becomes stable and the network dynamics converges to the vicinity of the stimulated pattern.

kept in mind that the stimulated pattern has its own basin of attraction and the network will naturally converge to a $m_\delta \neq 0$ phase if the initialization state is chosen close to the stimulated pattern.

The particular case of $T=0$ and an infinite number of stored patterns has a behavior closely related to the previous one, once the stored capacity introduces noise to the network in some sense similar to the thermal noise. The superpositions can be written in this case as

$$m_\nu(t+1) = \mathrm{erf}\left(\frac{m_\nu(t)}{\sqrt{2m_\delta(t)^2(1+h)^2+2\alpha}}\right), \qquad (17)$$

$$m_\delta(t+1) = \mathrm{erf}\left(\frac{m_\delta(t)(h+1)}{\sqrt{2m_\nu(t)^2+2\alpha}}\right). \qquad (18)$$

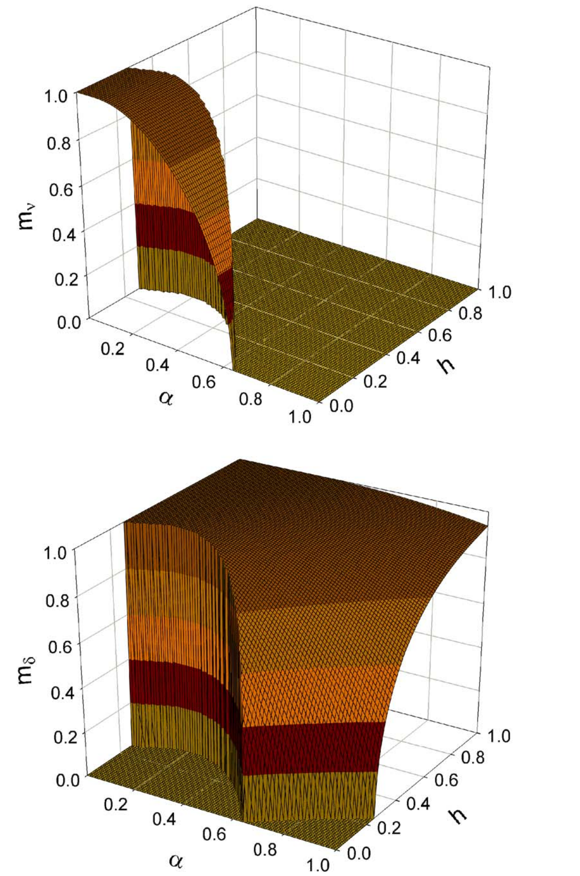

FIG. 4. (Color online) The $T=0$ superpositions m_ν and m_δ as functions of α and h for the case of orthogonal stimulus. The initial superpositions were taken to be $m_\nu(0)=0.99$ and $m_\delta(0)=0.01$. The storage capacity α has a role similar to the thermal noise.

The $T=0$ asymptotically stationary superpositions m_ν and m_δ as functions of α and h are shown in Fig. 4. Notice again that the initialization pattern ν can only be recognized by the neural network for small values of the stimulus field. The phase diagram in the $\alpha \times h$ plane for $T=0$ is shown in Fig. 5. The line separating the stable phase with $m_\delta=0$ and the nonrecognition phase is the same as the critical line for parallel stimulus. The dashed line represents the first-order transition between the phases with the stable state being the presented or the stimulated pattern. Here, the reentrant behavior is clearly depicted for stimulus fields slightly above the critical value $h_c \simeq 0.279$. The region of stability for the recognition of the presented pattern ν increases as the initial superposition with the stimulated state decreases. In the asymptotic limit of $m_\delta(0) \to 0$ it extends over the whole region of $\alpha < 2/\pi$.

For stimulus fields slightly above the first-order transition, the convergence towards the stationary state becomes slower due to the proximity of the frontier between the two basins of attraction. We have investigated the scaling behavior of this transient time by following the time evolution of the state superposition $m_\nu(t)$, as shown in Fig. 6. We have considered the representative case of $\alpha=0.25$ at which the first order transition occurs at $h^*=0.334$. The state superposition m_ν stays in a quasistationary state during the initial transient

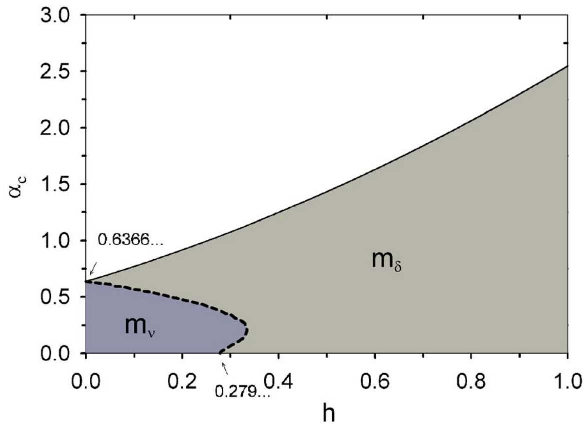

FIG. 5. (Color online) The $T=0$ phase diagram for the case of orthogonal stimulus with $m_\nu(0)=0.99$ and $m_\delta(0)=0.01$. The dashed line represents the discontinuous transition between the phase with $m_\nu \neq 0$ and the phase with $m_\delta \neq 0$. The reentrant behavior reflects the competing effects of the network dynamics and the stimulus field.

before the final crossover to the true stationary state with $m_\nu=0$. The duration of the initial transient diverges as one approaches the transition point. In the inset of Fig. 6 we depict the scaling behavior of the transient time. It exhibits a power-law scaling on the form $\tau \propto (h-h^*)^{-0.80}$.

We further computed the state superpositions as functions of T and α for a small value of the stimulus field $h=0.2$ (see Fig. 7). The discontinuous transition from the state with $m_\nu \neq 0$ to the state with $m_\delta \neq 0$ is followed by the continuous transition for the nonrecognition state. For very week fields (not shown) the initialization pattern is the one predominantly recognized, with the stimulated state being stable only on a short range of parameters. On the other hand, for large fields only the stimulated state can be recognized.

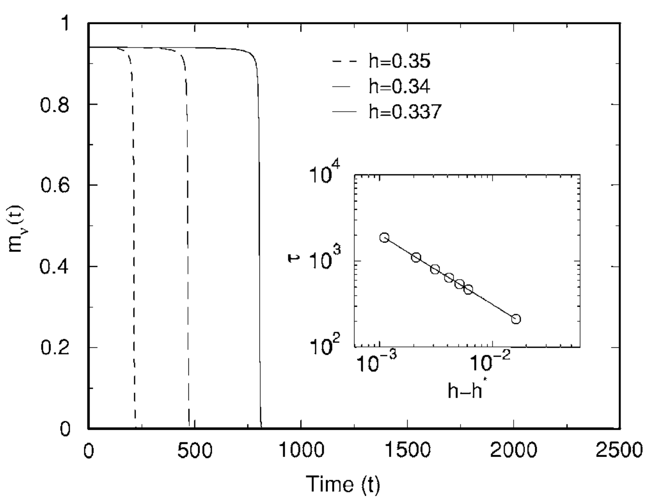

FIG. 6. Time evolution of the state superposition with the initialization pattern $m_\nu(t)$ at the vicinity of the first-order transition. The particular case of $\alpha=0.25$ for which $h^*=0.334$ is shown. For fields slightly above h^*, the system stays in a quasistationary state during a transient time before crossing over to the final stationary state. The inset exhibits the scaling behavior of the transient time which obeys $\tau \propto (h-h^*)^{-0.80}$.

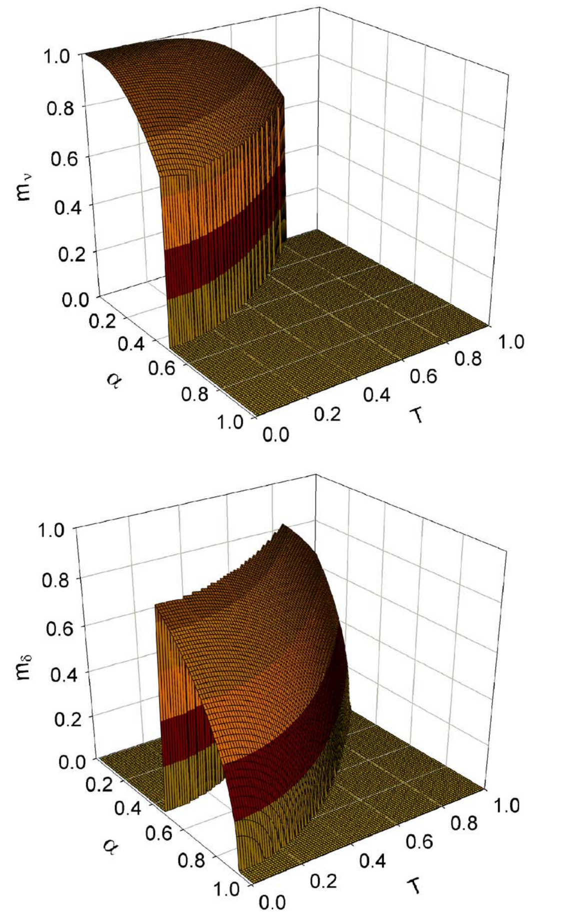

FIG. 7. (Color online) State superpositions as functions of T and α for an orthogonal stimulus field $h=0.2$. At low values of T and α, the network still recognizes the pattern strongly correlated to the initial configuration. Before the regime of no recognition, the network exhibits a phase that converges to the vicinity of the stimulated memory.

C. Numerical simulations at $T=0$

Numerical simulations have contributed to a deeper understanding of the behavior of dynamical models, besides being a fundamental tool to test analytical predictions based on approximated techniques. In the present section we compare the mean field analytical predictions obtained in the extreme dilution limit with numerical simulations performed at $T=0$. Due to the practical impossibility of achieving the limit of extreme dilution $C \ll \ln N$ for finite lattices and $C \gg 1$, we consider a less restrictive condition of $C \ll N$. Previous simulations have demonstrated that even with such less restrictive condition, the numerical results depict a good agreement with the extreme dilution mean-field results [18–20].

In order to employ the dynamic evolution of the local variables, the local field can be written in terms of the state superposition and the stored patterns. For the particular case in which the stimulus field acts over the pattern that is solely correlated to the initialization state, the dynamic equation reads

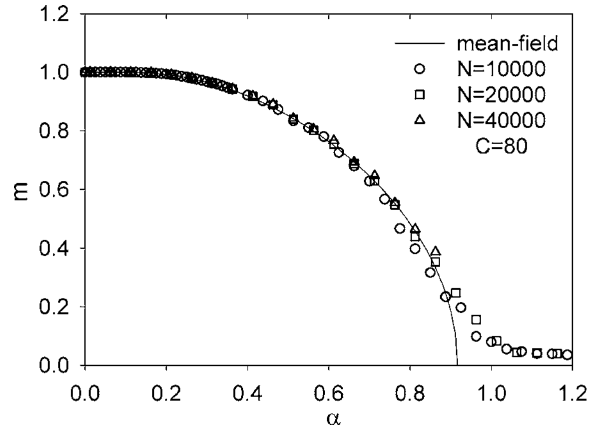

FIG. 8. State superposition m as a function of α for $T=0$ and parallel stimulus field $h=0.2$. Symbols corresponds to the numerical simulation results. The solid line is the analytic mean field prediction in the limit of extreme dilution. The agreement is remarkable considering that the extreme dilution condition is far from being fulfilled in the numerical simulation. N and C are explained in the text

$$S_i(t+1) = \mathrm{sgn}\left[\xi_i^\nu m_\nu(t)(h+1) + \frac{1}{C}\sum_{j=1}^{N}\sum_{\mu\neq\nu}^{p} C_{ij}\xi_i^\mu\xi_j^\mu S_j(t)\right]. \tag{19}$$

On the other hand, when the stimulus field acts in one out of two patterns correlated to the initialization state, the dynamic equation assumes the form

$$S_i(t+1) = \mathrm{sgn}\left[\xi_i^\nu m_\nu(t) + \xi_i^\delta(t) m_\delta(t)(h+1) + \frac{1}{C}\sum_{j=1}^{N}\sum_{\mu\neq\nu,\delta}^{p} C_{ij}\xi_i^\mu\xi_j^\mu S_j(t)\right]. \tag{20}$$

The above equations are valid in the absence of thermal noise. In our numerical simulations we computed the average superposition of the system state with the patterns ν and δ after reaching the stationary state. We averaged over 50 distinct realizations of the neural network, each one containing a new set of uncorrelated stored patterns and connectivity distribution. We used a parallel processing in networks with $N=10\,000$, 20 000, and 40 000 neurons and the average connectivity $C=80$. In all samples, we started with an initial configuration with superpositions $m_\nu(0)=0.99$ and $m_\delta(0)=0.01$ for orthogonal stimulus and $m_\nu(0)=1$ for parallel stimulus to be able to directly compare the numerical results with the analytical ones.

In Fig. 8, we present our numerical results for the case of parallel stimulus. We obtained an excellent agreement with the analytical prediction except at the vicinity of the transition point where finite-size effects round off the numerical data. This is a striking agreement once the extreme dilution condition is not fulfilled even for the largest network size considered. In Fig. 9 we depict our numerical data for the case of a stimulus field applied over a pattern that is orthogonal to the one strongly correlated to the initial state. We used $h=0.2$ for which the initialization pattern is stable for small capacities α (see Fig. 5). Our simulation results corroborate the analytical calculations in the limit of extreme dilution. A first order transition separates the phase with stable initialization pattern from that of stable stimulated pattern. As the capacity α is increased a continuous transition to a phase of no recognition ability takes place, equivalent to the one obtained for parallel stimulus.

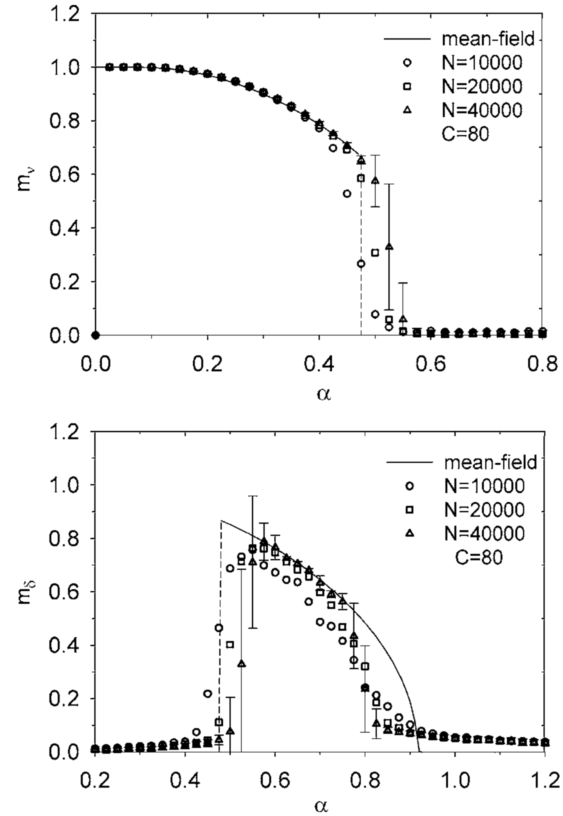

FIG. 9. State superpositions m_ν and m_δ as functions of α for $T=0$ and orthogonal stimulus field $h=0.2$. We considered $m_\nu(0)=0.99$ and $m_\delta(0)=0.01$. The numerical data corroborates the mean field results exhibiting also a first-order transition $m_\nu \rightarrow m_\delta$ followed by a continuous transition for a phase with no recognition ability. The simulation data are rounded off at the vicinity of the transition points due to finite-size corrections. N and C are explained in the text

IV. SUMMARY AND CONCLUSIONS

In summary, we studied the effects of a stimulus field on the recognition ability of neural networks. We took the basic Hopfield model in the ultradiluted regime to represent a network able to recognize patterns by associative memory. Following the procedure introduced by Derrida, Gardner, and Zippelius [14] we provided exact recurrence relations for the network state superposition. The stimulus field was introduced as an extra effective local field proportional to the state superposition with a particular stored pattern. One should note that the stimulus field enhances the recognition ability whenever it acts as a bias towards a pattern strongly correlated to the initial network configuration. In the limit of extreme dilution, we obtained analytical expressions for the state superposition and the phase diagram which were consistent with numerical simulations of finite networks. On the other hand, a stimulus field towards a pattern orthogonal to the one strongly correlated to the initial network configuration reduces the recognition ability of the initialization pattern. Actually strong stimulus fields can drive the network to recognize the stimulated pattern even for a slight superposition of the initialization configuration with the stimulated memory. We found a first-order transition between the phase with stable initial state and the phase with stable stimulated pattern, whose phase diagram presents a reentrant behavior due to the competition between the stimulus field and the recognition dynamics. Numerical simulations at $T=0$ have reproduced such first-order transition. As the simulations were performed in a diluted lattice far from the ultradiluted regime, we believe that the here reported analytical results shall remain valid for a much larger range of dilutions. It is important to point out that the presently introduced stimulus field does not act directly by broken network symmetry, i.e., its presence does not induce a finite state superposition (order parameter) in the regime of high temperatures and storage capacities. The network can still recognize a pattern strongly correlated to the one presented in the initial state even in the presence of a small bias towards an orthogonal configuration. This feature is also likely to occur in biological neural networks. We believe the present proposal represents one of the simplest ways to introduce this phenomenon in neural network modeling.

ACKNOWLEDGMENTS

We would like to thank S. G. Coutinho for fruitful comments and suggestions. This work was partially supported by the Brazilian research agencies CNPq and CAPES and by Alagoas state research agency FAPEAL.

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